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A SZEMERÉDI TYPE THEOREM FOR SETS OF POSITIVE DENSITY IN $\mathbf{R}^k$

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ABSTRACT

Let $k \geqq 2$ and A a subset of $\mathbf{R}^k$ of positive upper density. Let V be the set of vertices of a (non-degenerate) $(k-1)$-dimensional simplex. It is shown that there exists $l = l(A, V)$ such that A contains an isometric image of $l'.V$ whenever $l' > l$. The case $k = 2$ yields a new proof of a result of Katznelson and Weiss [4]. Using related ideas, a proof is given of Roth's theorem on the existence of arithmetic progressions of length 3 in sets of positive density.

1. Introduction

The following result has been obtained by Katznelson and Weiss [4].

THEOREM 1. *Whenever A is a subset of $\mathbf{R}^2$ with positive upper density, there is a number $l = l(A)$ such that $|x - y| = l'$ for some $x, y \in A$, fixing any $l' > l$. Recall that $A \subset \mathbf{R}^k$ has positive upper density provided*

$$\delta(A) \equiv \overline{\lim_R} \frac{|B(0,R) \cap A|}{|B(0,R)|} > 0$$

where $B(0,R) = \{x \in \mathbf{R}^k ; |x| < R\}$.

Their argument combines ergodic theory and measure theory. In the next section, a short proof will be given based on elementary harmonic analysis. This proof can be elaborated in order to get the result mentioned above:

THEOREM 2. *Assume $A \subset \mathbf{R}^k$, $\delta(A) > 0$ and V a set of k points spanning a $(k-1)$-dimensional hyperplane. There exists some number l such that A contains an isometric copy of $l'.V$ whenever $l' > l$.*

REMARKS. (a) Theorem 2 is of the same nature as the generalizations of

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Szemerédy's theorem [7] obtained in [3] (see also [2]). More precisely, the dilations are replaced by rotations. Although the method presented here requires an increasing dimension, the exact rôle of the dimension k does not seem well understood yet.

(b) The following simple example clarifies the necessity of the non-degeneracy hypothesis on the set V. Let $V=\{-1,0,1\}$ and $A=\{x\in\mathbf{R}^k;|x|^2\in[0,\frac{1}{10}]+\mathbf{Z}_+\}$. Clearly $\delta(A)>0$. Assume now $x\in A$ and $y\in\mathbf{R}^k$, $|y|=t$ satisfying $x+y\in A$ and $x-y\in A$. Then

$$2t^2=2|y|^2=|x+y|^2+|x-y|^2-2|x|^2\in[-\tfrac{1}{5},\tfrac{1}{5}]+\mathbf{Z}_+$$

implying the existence of some $k\in\mathbf{Z}_+$ s.t.

$$\left|t-\sqrt{\frac{k}{2}}\right|<\frac{1}{5\sqrt{k}}.$$

Consequently, there are arbitrary large values of l such that A does not contain an isometric copy of $l.V$. This example permits several variations.

It is easily seen that Theorems 1 and 2 result from the following "compact" version.

PROPOSITION 3. *Let V be as in Theorem 2,* diam $V<1$. *Let $A\subset[0,1]^k$, $|A|>\varepsilon$ and $0<t_j<1$ a sequence satisfying $t_{j+1}<\frac{1}{2}t_j$. Then there exists $j\leqq J(\varepsilon,V)$ such that A contains an isometric image of $t_j.V$. In fact, for $t=t_j$*

$$\int_{\mathbf{R}^k}\int_{\mathrm{SO}(k)} f(x)f(x+t0a_1)\cdots f(x+t0a_{k-1})dxdO>\tfrac{1}{2}\varepsilon^k \tag{1}$$

where $f=\chi_A$, $V=\{0,a_1,\ldots,a_{k-1}\}$ and dO refers to the normalized invariant measure on the orthogonal group SO(k).

For the sake of clarity, the case $k=2$ will be handled separately. The complete proof of Proposition 3 is given in section 3 of this paper. The last section is an appendix in which it is shown how a new proof of Roth's theorem (see [5]) can be obtained using similar ideas. The letters $0<c,C<\infty$ denote numerical constants

2. A proof of the Katznelson–Weiss theorem

As usual $\hat{F}(\xi)=\int_{\mathbf{R}^k}F(x)e^{-2\pi i\langle x,\xi\rangle}dx$ stands for the Fourier transform. In case $k=2$, the left member of (1) becomes

$$\int\int f(x)f(x+ty)dx\sigma(dy)=\int\hat{f}(\xi)\hat{f}(-\xi)\hat{\sigma}(t\xi)d\xi=\int|\hat{f}(\xi)|^2\hat{\sigma}(t|\xi|)d\xi \tag{2}$$

where σ denotes the normalized arc-length measure of the unit circle. Thus

$$|\hat{\sigma}(\xi)| \leqq C|\xi|^{-\frac{1}{2}} \quad \text{and} \quad |1-\hat{\sigma}(\xi)| < C|\xi|. \tag{3}$$

Also, by definition of f

$$|\hat{f}(\xi)-\hat{f}(0)| \leqq 2\pi \int_A |\langle x,\xi\rangle|\,dx; \qquad |\hat{f}(\xi)-|A|| < C|\xi||A|.$$

Hence, for $\delta > t$ to be specified later, as a consequence of (3)

$$\int |\hat{f}(\xi)|^2\hat{\sigma}(t|\xi|)d\xi = \left\{\int_{[|\xi|\leqq \delta t^{-1}]} + \int_{[\delta t^{-1}<|\xi|<\delta^{-1}t^{-1}]} + \int_{[|\xi|>\delta^{-1}t^{-1}]}\right\}|\hat{f}(\xi)|^2\hat{\sigma}(t\xi)d\xi$$

$$\geqq \frac{1}{2}\int_{[|\xi|\leqq \delta t^{-1}]} |\hat{f}(\xi)|^2 d\xi - \int_{[\delta t^{-1}<|\xi|<\delta^{-1}t^{-1}]} |\hat{f}(\xi)|^2 d\xi - C\delta^{1/2}\int |\hat{f}(\xi)|^2 d\xi$$

$$\geqq C_1|A|^2 - C\delta^{1/2}|A| - \int_{[\delta t^{-1}<|\xi|<\delta^{-1}t^{-1}]} |\hat{f}(\xi)|^2 d\xi.$$

Assume $\delta \ll |A|^2$. It is clear that there exists some

$$j \leqq C\left(\log\frac{1}{\delta}\right)\varepsilon^{-1} \sim \frac{1}{\varepsilon}\log\frac{1}{\varepsilon}$$

satisfying

$$\int_{[\delta t_j^{-1}<|\xi|<\delta^{-1}t_j^{-1}]} |\hat{f}(\xi)|^2 d\xi < \tfrac{1}{3}C_1\varepsilon^2$$

and therefore

$$\iint f(x)f(x+t_jy)dx\sigma(dy) \geqq \tfrac{1}{2}C_1|A|^2. \qquad \text{QED}$$

REMARK. Combined with the results on the spherical maximal function in the plane, Theorem 1 can be improved as follows:

THEOREM 1′. *If $A \subset \mathbf{R}^2$, $\delta(A) > 0$, there exists $l = l(A)$ such that whenever $l_1 > l$ there is a point $x \in A$ fulfilling the condition*

$$\{|x-y|; y \in A\} \supset [l, l_1].$$

Denote P_t the Poisson-semigroup kernel on $\mathbf{R}^k$. Thus $\hat{P}_t(\xi) = e^{-t|\xi|}$. In general, let $K_t(x) = t^{-k}K(t^{-1}x)$ satisfying $\hat{K}_t(\xi) = \hat{K}(t\xi)$.

The key estimate of [1] related to the planar spherical maximal operator can be formulated as follows:

PROPOSITION 1. *For $p>2$, there are constants $C(p)<\infty$ and $\alpha(p)>0$ satisfying*

$$\left\| \max_{s \geqq t_0} |[f-(f*P_t)]*\sigma_s| \right\|_p \leqq C(p)\left(\frac{t}{t_0}\right)^{\alpha(p)} \|f\|_p, \qquad t_0>t. \tag{4}$$

Similarly, as in proving Theorem 1, the negation of Theorem 1′ leads to a subset A of $[0,1]^2$, $|A|>\varepsilon$ and a sequence of positive numbers

$$s_1>t_1>s_2>t_2>\cdots>s_J>t_J$$

where J can be taken arbitrarily large, satisfying the properties

$$s_{j+1}<\tfrac{1}{2}t_j \tag{5}$$

and

$$x \in A \cap [s_j, 1-s_j]^2 \Rightarrow \sup_{s_j>t>t_j} [(1_R-f)*\sigma_t]=1 \qquad (f=1_A \quad \text{and} \quad R=[0,1]^2).$$

Hence, we may write for a fixed $\tau>0$ and choosing $j<J$ large enough

$$\int f \sup_{s_j>t>t_j} [(1_R-f)*\sigma_t]<(1-\tau)\int f. \tag{6}$$

Fix $\delta>0$. As a consequence of (4), we may write

$$\begin{aligned} &\left\| \sup_{s_j>t>t_j} [(1_R-f)*\sigma_t] - \sup_{s_j>t>t_j} [(1_R-f)*P_{\delta t_j}*\sigma_t] \right\|_1 \\ &\leqq \left\| \sup_{t>t_j} |[f-f*P_{\delta t_j}]*\sigma_t| \right\|_p + \tau < C\delta^\alpha+\tau. \end{aligned} \tag{7}$$

For $t<s_j$, also

$$|[(1_R-f)*P_{\delta^{-1}s_j}*\sigma_t](x)-[(1_R-f)*P_{\delta^{-1}s_j}](x)| \leqq \|P_{\delta^{-1}s_j}-(P_{\delta^{-1}s_j}*\sigma_t)\|_1<C\delta. \tag{8}$$

Thus, again by (4), using (6), (7), (8)

$$\begin{aligned} &C\|[(1_R-f)*P_{\delta^{-1}s_j}]-[(1_R-f)*P_{\delta t_j}]\|_p \\ &\geqq \int f \sup_{s_j>t>t_j} \{[(1_R-f)*P_{\delta t_j}*\sigma_t]-[(1_R-f)*P_{\delta^{-1}s_j}*\sigma_t]\} \\ &\geqq (1-\tau)\int f - \int f[(1_R-f)*P_{\delta^{-1}s_j}] - C\delta^\alpha - \tau \\ &\geqq -2\tau\int f + \left(\int f\right)^2 - C\delta^\alpha - \tau \\ &\geqq (\varepsilon-2\tau)\varepsilon - C\delta^\alpha - \tau. \end{aligned}$$

Taking τ, δ small enough and J sufficiently large, a contradiction follows. Indeed, if $t_{j+1} < \frac{1}{2} t_j$, then for $2 \leqq p \leqq \infty$

$$\left\{ \sum_j \| (f * P_{t_{j+1}}) - (f * P_{t_j}) \|_p^p \right\}^{1/p} \leqq C \| f \|_p.$$

This completes the proof of Theorem 1′.

3. Proof of Theorem 2 in the general case

Let $V = \{0, a_1, a_2, \ldots, a_{k-1}\}$ be non-degenerated. Simple invariance arguments show that the left member of (1) may be rewritten as

$$\int f(x) f(x + ty_1) f(x + ty_2) \cdots f(x + ty_{k-1}) \sigma^{(k-1)}(dy_1) \sigma_{y_1}^{(k-2)}(dy_2) \cdots \sigma_{y_1, \ldots, y_{k-2}}^{(1)}(dy_{k-1})$$

(9)

where $\sigma^{(j)}_{y_1, \ldots, y_{k-j-1}}$ is the average on a (j)-dimensional sphere in $\mathbf{R}^k$ dependent on the points $y_1, \ldots, y_{k-j-1}$ already fixed and on V. We will use the estimate

$$(10) \qquad | \widehat{\sigma^{(j)}_{y_1, \ldots, y_{k-j-1}}}(\xi) | \leqq C_V [1 + \operatorname{dist}(\xi, [y_1, \ldots, y_{k-j-1}])]^{-j/2}$$

which is a consequence of the decay at infinity of the Fourier transform of the j-sphere in $\mathbf{R}^{j+1}$. Denote $G_{k,m}$ $(m < k)$ the Grassmannian of m-dimensional subspaces of $\mathbf{R}^k$ endowed with the normalized Haar-measure.

LEMMA 1. *For $m < k$*

$$(11) \qquad \int_{\mathbf{R}^k} \int_{G_{k,m}} [\operatorname{dist}(\xi, F) + 1]^{-\rho} | \hat{f}(\xi) |^2 (1 - e^{-\delta|\xi|})^2 d\xi dF < C_k (\delta + \delta^{\rho/2}) \| f \|_2^2.$$

PROOF. Estimate the left member of (11) as

$$C\delta \| f \|_2^2 + C \left\{ \sup_{|\xi| > \delta^{-1/2}} \int_{G_{k,m}} [\operatorname{dist}(\xi, F) + 1]^{-\rho} dF \right\} \| f \|_2^2.$$

PROOF OF THEOREM 2. Denote for simplicity

$$d\Omega_j(y_1, \ldots, y_j) = \sigma^{(k-1)}(dy_1) \sigma_{y_1}^{(k-2)}(dy_2) \cdots \sigma_{y_1, \ldots, y_{j-1}}^{(k-j)}(dy_j).$$

Fix $\delta > 0$ and compare the expressions

$$(12) \qquad \int f(x) f(x + ty_1) \cdots f(x + ty_{k-2}) f(x + ty_{k-1}) dx d\Omega_{k-1}(y_1, \ldots, y_{k-1})$$

and

$$(13)\quad \int f(x)f(x+ty_1)\cdots f(x+ty_{k-2})(f*P_{\delta t})(x+ty_{k-1})dx\,d\Omega_{k-1}(y_1,\ldots,y_{k-1})$$

which difference can be estimated as

$$|(12)-(13)| \leqq \int \|[f-(f*P_{\delta t})]*[\sigma^{(1)}_{y_1,\ldots,y_{k-2}}]_t\|_2\, d\Omega_{k-2}(y_1,\ldots,y_{k-2})$$

or by Parseval's identity, using (10), (11), as

$$(14)\quad C_V\left\{\int_{\mathbf{R}^k}\int_{G_{k,k-2}} |\hat{f}(\xi)|^2[1-e^{-\delta t|\xi|}]^2\{1+\operatorname{dist}(t\xi,F)\}^{-1/2}d\xi\,dF\right\}^{1/2} \leqq C_V\delta^{1/4}\|f\|_2.$$

Next we compare the expressions

$$(15)\quad \int f(x)(f*P_{\delta^{-1}t})(x)f(x+ty_1)\cdots f(x+ty_{k-2})dx\,d\Omega_{k-1}(y_1,\ldots,y_{k-1})$$

and

$$(16)\quad \int f(x)f(x+ty_1)\cdots f(x+ty_{k-2})(f*P_{\delta^{-1}t})(x+ty_{k-1})dx\,d\Omega_{k-1}(y_1,\ldots,y_{k-1}),$$

which difference is simply majorated by

$$\begin{aligned}&\sup_{|y|<1}\|(f*P_{\delta^{-1}t})(x)-(f*P_{\delta^{-1}t})(x+ty)\|_{L^2(dx)}\\ (17)\qquad &\leqq \sup_{|y|<1}\left\{\int |\hat{f}(\xi)|^2|1-e^{2\pi i(ty,\xi)}|^2 e^{-\delta^{-1}t|\xi|}d\xi\right\}^{1/2}\\ &< C\delta\|f\|_2.\end{aligned}$$

Collecting estimates, it now follows that

$$\begin{aligned}|(12)-(15)| &\leqq |(12)-(13)|+|(15)-(16)|+|(13)-(16)|\\ &\leqq C_V\delta^{1/4}\|f\|_2+\|(f*P_{\delta^{-1}t})-(f*P_{\delta t})\|_2.\end{aligned}$$

In the expression (15)

$$(15)=\int f(x)(f*P_{\delta^{-1}t})(x)f(x+ty_1)\cdots f(x+ty_{k-2})dx\,d\Omega_{k-2}(y_1,\ldots,y_{k-2}),$$

the variable y_{k-1} does not appear any more. We treat (15) the same way as (12) where y_{k-2} plays the rôle of y_{k-1}. Thus defining

$$(17)=\int f(x)(f*P_{\delta^{-1}t})^2(x)f(x+ty_1)\cdots f(x+ty_{k-3})dx\,d\Omega_{k-3}(y_1,\ldots,y_{k-3})$$

similar computations give

$$|(15)-(17)| \leqq C_V(\delta+\delta^{1/2})\|f\|_2 + \|(f * P_{\delta^{-1}t}) - (f * P_{\delta t})\|_2.$$

Iteration of the procedure yields

$$\left|(12) - \int f(x)(f * P_{\delta^{-1}t})^{k-1}(x)dx\right|$$

(18)
$$\leqq C_V\left(k\delta + \sum_{\delta=1}^{k-1} \delta^{r/4}\right)\|f\|_2 + k\,\|(f * P_{\delta^{-1}t}) - (f * P_{\delta t})\|_2.$$

Further

$$\varepsilon^k \leqq \int (f * P_{\delta^{-1}t})^k$$

$$\leqq \int (f * P_{\delta^{-1}t})^{k-1}(f * P_{\delta t}) + \|(f * P_{\delta t}) - (f * P_{\delta^{-1}t})\|_2$$

$$\leqq \int f.(f * P_{\delta^{-1}t})^{k-1} + \|(f * P_{\delta^{-1}t})^{k-1} - [P_{\delta t} * (f * P_{\delta^{-1}t})^{k-1}]\|_2$$

$$+ \|(f * P_{\delta t}) - (f * P_{\delta^{-1}t})\|_2$$

where the second term is dominated by

$$\sqrt{2}\left\{\int (f * P_{\delta^{-1}t})^{2(k-1)} - \int [P_{\delta t/2} * (f * P_{\delta^{-1}t})^{k-1}]^2\right\}^{1/2}$$

$$\leqq \sqrt{2}\left\{\int (f * P_{\delta^{-1}t})^{2(k-1)} - \int (f * P_{\delta^{-1}t} * P_{\delta t/2})^{2(k-1)}\right\}^{1/2}$$

$$\leqq Ck\|(f * P_{\delta^{-1}t}) - (f * P_{\delta^{-1}t} * P_{\delta t/2})\|_2$$

$$\leqq Ck\delta^2\|f\|_2.$$

Therefore, as a consequence of (18) and previous computation

$$(12) \geqq \varepsilon^k - C_V k\delta^{1/4}\|f\|_2 - \|(f * P_{\delta t}) - (f * P_{\delta^{-1}t})\|_2(k+1).$$

Taking suitable $t \in \{t_1 > t_2 > \cdots > t_J\}$ ($t_{j+1} < \frac{1}{2}t_j$), we may dominate

$$\|(f * P_{\delta t}) - (f * P_{\delta^{-1}t})\|_2 \leqq \frac{C}{J}\left(\log\frac{1}{\delta}\right)\|f\|_2$$

so that

$$(12) \geqq \varepsilon^k - C_V k\left(\delta^{1/4} + J^{-1}\left(\log\frac{1}{\delta}\right)\right)\sqrt{\varepsilon} > \tfrac{1}{2}\varepsilon^k$$

for an appropriate choice of δ and J. This completes the proof.

4. Appendix: A proof of Roth's theorem on arithmetic progressions of length 3

Let G be a compact Abelian group and $\Gamma = \hat{G}$ the dual group.

THEOREM 3. *Given $\varepsilon > 0$, there exists $\varepsilon' = \varepsilon'(\varepsilon)$ such that whenever f is a function on G, $0 \leqq f \leqq 1$ and $\int_G f(x)dx > \varepsilon$, then*

$$\iint_{G\times G} f(x)f(x+y)f(x+2y)dxdy > \varepsilon'. \tag{1}$$

Applying the result to a finite cyclic group $G = \mathbf{Z}/N\mathbf{Z}$ (taking N large enough) and $f = \chi_S$ ($S \subset G$, $|S| > \varepsilon$) yields Roth's theorem ([5]).

The proof is based on two lemmas:

LEMMA 2. $|\iint f_1(x)f_2(x+y)f_3(x+2y)K(y)dxdy| \leqq \|K\|_{A(G)} \prod_{i=1}^{3} \|\hat{f}_i\|_\infty^{1/3} \|f_i\|_2^{2/3}$.

PROOF.

$$\begin{aligned}&|\langle f_1, \int f_2(\cdot + y)f_3(\cdot + 2y)K(y)dy\rangle|\\ &\leqq \|\hat{f}_1\|_\infty \|\int f_2(\cdot + y)f_3(\cdot + 2y)K(y)dy\|_{A(G)}\end{aligned}$$

and the second factor is dominated by $\|K\|_{A(G)}\|f_2\|_2\|f_3\|_2$. Reversing the rôle of f_1, f_2 and making the product gives the estimate.

LEMMA 3 (Bozejko–Pelczynski theorem on invariant approximation, cf. [8]). *Given a finite subset Λ of Γ and $\tau > 0$, there exists a kernel K satisfying*

(i) $K \geqq 0$, $\hat{K} \geqq 0$ *and* $\hat{K}(0) = 1$,

(ii) $|\hat{K}(\gamma) - 1| < \tau$ *for* $\gamma \in \Lambda$,

(iii) $|\text{supp}\, \hat{K}| < N(|\Lambda|, \tau)$.

PROOF OF THEOREM 3. Let f be as in Theorem 1. Combining Lemmas 1 and 2, it follows that given a kernel K with $\hat{K}$ finitely supported, there exists K' satisfying (i) of Lemma 2 and

$$|K - (K * K')| < \tau, \tag{2}$$

$$\begin{aligned}&\left|\iint f(x)f(x+y)f(x+2y)K(y)dxdy\right.\\ &\left.-\iint (f*K')(x)(f*K')(x+y)(f*K')(x+2y)K(y)dxdy\right| < \tau,\end{aligned} \tag{3}$$

$$|\text{supp}\, \hat{K}'| < N'(|\text{supp}\, \hat{K}|, \|\hat{K}\|_\infty, \tau). \tag{4}$$

Take $K_0 = 1$. Previous considerations and an inductive construction lead to a sequence $\{K_i\}_{0 \leqq i < I}$ satisfying (i) of Lemma 2 (I is a positive integer of size $\sim \varepsilon^{-3}$).

Denote $f_i = f * K_i$. By (2), $|f_i - (f_i * K_{i+1})| < \tau$. Thus

$$\|f_{i+1} - f_i\|_2^2 = \|f_{i+1}\|_2^2 + \|f_i\|_2^2 - 2\langle f_i, f_{i+1}\rangle \leqq \|f_{i+1}\|_2^2 + \|f_i\|_2^2 - 2\langle f_i, f\rangle + 2\tau$$
$$\leqq \|f_{i+1}\|_2^2 - \|f_i\|_2^2 + 2\tau$$

and summation shows the existence of some $1 \leqq i \leqq I$ fulfilling

$$\|f_{i+1} - f_{i-1}\|_1 < 4\tau + 2I^{-1}$$

and hence

$$\tag{5} \left| \iint f_{i+1}(x) f_{i+1}(x+y) f_{i+1}(x+2y) K_i(y)\, dx dy - \iint f_{i-1}(x) f_{i-1}(x+y) f_{i-1}(x+2y) K_i(y)\, dx dy \right| < 12\tau + 6.$$

Assume (1) does not hold. From (3) and the construction ($K = K_i$, $K' = K_{i+1}$) it now follows from (5) that

$$\tag{7} \left| \iint f_{i-1}(x) f_{i-1}(x+y) f_{i-1}(x+2y) K_i(y)\, dx dy \right| < 13\tau + 6I^{-1} + \varepsilon' \|K_i\|_\infty.$$

Also, for $\gamma = 1, 2$

$$\tag{8} \begin{aligned} &\iint |f_{i-1}(x+\gamma y) - f_{i-1}(x+(\gamma-1)y)|\, K_i(y)\, dx dy \\ &\leqq \left\{ \iint |f_{i-1}(x+y) - f_{i-1}(x)|^2 K_i(y)\, dx dy \right\}^{1/2} \\ &= \sqrt{2}\,(\|f_{i-1}\|_2^2 - \langle f_{i-1}, f_{i-1} * K_i\rangle)^{1/2} \\ &< 4\sqrt{\tau} \end{aligned}$$

which permits us to replace in the left member of (7) $f_{i-1}(x+y), f_{i-1}(x+2y)$ by $f_{i-1}(x)$. Hence

$$\varepsilon^3 \leqq \left(\int_G f \right)^3 \leqq \iint f_{i-1}(x)^3 K_i(y)\, dx dy < 16\tau + 6I^{-1} + \varepsilon' \|K_i\|_\infty$$

giving a lower bound on ε'.

Remark. It follows, for instance, from the construction of Salem and Spencer (see [6], p. 252) that $\varepsilon'(\varepsilon)$ is not a polynomial function of ε in Theorem

3. However, there exist methods providing better bounds than results from the previous argument.

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